

Generalization of entanglement to convex operational theories: Entanglement relative to a subspace of observables

Howard Barnum¹, Gerardo Ortiz¹, Rolando Somma¹, Lorenza Viola²

Abstract

We define what it means for a state in a convex cone of states on a space of observables to be *generalized-entangled* relative to a subspace of the observables, in a general ordered linear spaces framework for operational theories. This extends the notion of ordinary entanglement in quantum information theory to a much more general framework. Some important special cases are described, in which the distinguished observables are subspaces of the observables of a quantum system, leading to results like the identification of generalized unentangled states with Lie-group-theoretic coherent states when the special observables form an irreducibly represented Lie algebra. Some open problems, including that of generalizing the semigroup of local operations with classical communication to the convex cones setting, are discussed.

KEY WORDS: Entanglement, convexity, ordered linear spaces, operational theories, observables.

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1. INTRODUCTION

Entanglement is a distinctively quantum phenomenon whereby a pure state of a composite quantum system may no longer be determined by the states of its constituent subsystems (Schrödinger, 1935). Entangled pure states are those that have *mixed* subsystem states. To determine an entangled state requires knowledge of the correlations between the subsystems. As no pure state of a classical system can be correlated, such correlations are intrinsically non-classical, as strikingly manifested by the possibility of violating local realism and Bell's inequalities Bell (1993). In the science of quantum information processing (QIP), entanglement is regarded as the defining resource for quantum communication, as well as an essential feature needed for unlocking the power of quantum computation.

The standard definition of quantum entanglement requires a preferred partition of the overall system into subsystems— that is, mathematically, a factorization of the Hilbert space as a tensor product. Even within quantum mechanics, there are motivations for going beyond such subsystem-based notions of entanglement. Whenever indistinguishable particles are sufficiently close to each

¹ Los Alamos National Laboratory, Los Alamos, NM 87545

² Department of Physics and Astronomy, Dartmouth College, 6127 Wilder Laboratory, Hanover, NH 03755

other, quantum statistics forces the accessible state space to be a proper subspace of the full tensor product space, and exchange correlations arise that are not a usable resource in the conventional QIP sense. Thus, the natural identification of particles with preferred subsystems becomes problematic. Even if a distinguishable-subsystem structure may be associated to degrees of freedom different from the original particles (such as a set of position or momentum modes, as in Zanardi (2002)), inequivalent factorizations may occur on the same footing. Entanglement-like notions not tied to modes have been proposed for bosons and fermions (Eckert *et al.*, 2002). However, the introduction of quasiparticles, or the purposeful transformation of the algebraic language used to analyze the system (Batista and Ortiz, 2001; Batista *et al.*, 2002), may further complicate the choice of preferred subsystems.

In this paper, we review and further develop *generalized entanglement* (GE) introduced in Barnum *et al.* (2003b), which incorporates the entanglement settings introduced to date in a unifying framework. In quantum-mechanical settings, the key idea behind GE is that entanglement is an *observer-dependent concept*, whose properties are determined by the expectations of a *distinguished subspace of observables* of the system of interest, without reference to a preferred subsystem decomposition. Distinguished observables may represent, for instance, a limited means of manipulating and measuring the system. Standard entanglement is recovered when these means are restricted to arbitrary *local* observables acting on subsystems. The central idea is to generalize the observation that standard entangled pure states are precisely those that look mixed to local observers.

The most fundamental aspects of this notion of GE make use only of the convex structures of the spaces of quantum states and observables. Therefore it is also applicable in contexts much broader than that of quantum systems with distinguished subspaces of observables. It may be formulated within general convex frameworks, based on ordered linear spaces or the closely related notion of convex effect algebras, suitable for investigating the foundations of quantum mechanics and related physical theories (cfr. Beltrametti and Bugajski (1997) and references therein). While commenting on physically motivated special cases, we will concentrate on this general setting in the present paper. Though we make no major advances over Barnum *et al.* (2003b) and Barnum *et al.* (2003a), new material here includes Theorem 3.4 which gives another characterization of the convex cones framework, in terms of restriction to a subspace of observables, and more detailed investigation of the distinguished quantum observables subspace. This includes the introduction of the *unique preimage property* (Def. 3.6) and the relationship between the quadratic purity measure, generalized entanglement, and the UPIP in this context, notably Problems 3.7, 4.8, and 4.9, and Propositions 4.5 and 4.6.

Two sets of articles contain related ideas. The first originated in the context of C^* and von Neumann algebras, for example in Connes *et al.* (1987), where

the dynamical entropy of automorphisms of algebras, intended to generalize the Kolmogorov-Sinai dynamical entropy, is defined — using a notion of entropy of a state’s restriction to a subalgebra introduced in Narnhofer and Thirring (1985). These ideas were further developed with special attention to finding optimal decompositions for the convex roof construction of entropy relative to a subalgebra, and applied to quantum information concepts such as quantum parameter estimation and the entanglement of formation. See e.g. Benatti (1996); Uhlmann (1998); Benatti *et al.* (1996); Benatti and Narnhofer (1998); Benatti *et al.* (2003). The association of subsystems, whether physical or “virtual”/“encoded,” of a quantum system with associative subalgebras appeared in in a second set of articles Knill *et al.* (2000); Filippo (2000); Viola *et al.* (2001); Zanardi (2001); this association was recently revisited, and examples collected, in Zanardi *et al.* (2004). Note, however, that these latter articles were not directly concerned with the *extremality properties of reduced states* which form the basis of our GE notion. Also, in both sets of articles, the context of subalgebras, whether C^* , von Neumann, or associative, is considerably more restrictive than the general context we work in here, except for the fact that Benatti, Connes, Narnhofer, Thirring, and Uhlmann often include and are sometimes primarily interested in infinite-dimensional algebras, whereas we focus here exclusively on the finite-dimensional setting.

2. MATHEMATICAL BACKGROUND

For background on cones and convexity, we highly recommend the text by Barvinok (2002), or the short introductory portion of Hilger *et al.* (1989); however, the summary we give here should suffice for what follows.

2.1 Definition A *positive cone* is a proper subset K of a real vector space V closed under multiplication by nonnegative scalars. It is called *regular* if it is (a) convex (equivalently, closed under addition: $K + K = K$), (b) generating ($K - K = V$, equivalently K linearly generates V), (c) pointed ($K \cap -K = \{0\}$, so that it contains no non-null subspace of V), and (d) topologically closed (in the Euclidean metric topology, for finite dimension). In the remainder of this paper, “vector space” and “linear space” will mean *finite-dimensional* vector space, “cone” will mean a regular cone in a finite-dimensional vector space, unless otherwise stated.

A cone K induces a partial order \geq_K on V , defined by $x \geq_K y := x - y \in K$. It is “linear-compatible”: inequalities can be added, and multiplied by positive scalars. If one removes the requirement that the cones be generating, cones are in one-to-one correspondence with linear-compatible partial orderings. A pair $\langle V, \succeq \rangle$ of a linear space and a distinguished such ordering is called an *ordered linear space*. The categories of real linear spaces with distinguished cones and partially ordered real linear spaces are equivalent.

Note that the intersection of the interior of a generating cone with a subspace

is (if not equal to $\{0\}$) a (*non-closed* but otherwise regular) cone that generates the subspace. When a cone or other set is said to generate a linear space, it does so via linear combination. When a set is said to generate a cone, it does so via positive linear combination. We will use the notation \dot{C} , for the set $C - \{0\}$.

By an *extremal* state in a convex set of states, we mean the usual convex-set notion that a point x is extremal in a convex set S if (and only if) it cannot be written as a nontrivial convex combination $x = \lambda_1 x_1 + \lambda_2 x_2$ of points x_1, x_2 in S . (Convex combination means $\lambda_i \geq 0, \lambda_1 + \lambda_2 = 1$, and nontrivial means $\lambda_i \neq 0, x_1 \neq x_2$). We sometimes use the physics term *pure state* for an extremal point in a convex set of states, but for clarification we emphasize that when this convex set is the set of all quantum states on some Hilbert space, the term “pure state” in the present paper refers to a projector $\pi := |\psi\rangle\langle\psi|$, and not to a vector $|\psi\rangle$ in the underlying Hilbert space. We write Extr S for the set of extremal points of a convex set S .

A *ray* belonging to a cone K is a set R such that there exists an $x \in K$ for which $R = \{\lambda x : \lambda \geq 0\}$, i.e. it is the set of all nonnegative scalar multiples of some element of the cone. An *extreme ray* in K is a ray R such that no $y \in R$ can be written as a convex (or equivalently, positive) combination of elements of K that are not in R . The topological closure condition guarantees, through an easy but not trivial argument using the Krein-Milman theorem, that a (regular) cone is convexly (equivalently, positively) generated by its extreme rays. We’ll say a point is extremal in a cone if it belongs to an extreme ray of the cone; note that such points are not usually extremal in the convex set sense, although the cone is a convex set; the only point in a cone extremal in the convex set sense is zero.

The dual vector space V^* for real V is the space of all linear functionals from V to \mathbb{R} ; the dual cone $C^* \subset V^*$ of the cone $C \subset V$ is the set of such linear functionals which are nonnegative on C . $\lambda \in V^*$ is said to separate C from $-C$ if $\lambda(x) \geq 0$ for all nonzero $x \in C$. For $\alpha \in V^*$, $x \in V$, we write the value of α on x as $\alpha[x]$, rather than $\alpha(x)$. The adjoint $\phi^* : V_2^* \rightarrow V_1^*$ of a linear map $\phi : V_1 \rightarrow V_2$ is defined by

$$\phi^*(\alpha)[x] = \alpha[\phi(x)] , \quad (1)$$

for all $\alpha \in V_2^*, x \in V_1$. The following proposition is easily (but instructively) verified.

2.2 Proposition *Let C_i be a cone in V_i for $i = 1, 2$, and let $\phi(C_1) \subseteq C_2$. Then $\phi^*(C_2^*) \subseteq C_1^*$.*

We will also use the following:

2.3 Proposition *Let C_i be a cone in V_i for $i = 1, 2$, and let $\phi(C_1) = C_2$. Then $\phi^*(C_2^*) \subseteq C_1^*$ and ϕ^* is one-to-one.*

Proof: Let $\eta_1, \eta_2 \in C_2^*$, and $\eta_1 \neq \eta_2$. $\eta_1 \neq \eta_2$ is equivalent to the existence of y in C_2 such that $\eta_1[y] \neq \eta_2[y]$. By the assumption that ϕ maps C_1 onto C_2 , there is an $x \in C_1$ such that $\phi(x) = y$; thus $\eta_1[\phi(x)] \neq \eta_2[\phi(x)]$. By the definition of ϕ^* , this implies that $\phi^*(\eta_1)[x] \neq \phi^*(\eta_2)[x]$, which implies that $\phi^*(\eta_1) \neq \phi^*(\eta_2)$. \square

3. GENERALIZED ENTANGLEMENT

We now introduce GE of states in a convex set of states given by the intersection \hat{C} of an affine “normalization” plane $\{x : \lambda(x) = \alpha\}$ (for a fixed α , which we’ll take to be one) with a cone C of “unnormalized states.” This GE is a relative notion: states are entangled or unentangled relative to another such state-set \hat{D} , and a choice of normalization-preserving map of the first state-set onto the second, which generalizes the notion of computing the reduced density matrices of a bipartite system. To fix ideas, note that in the case where C is supposed to represent states on a finite dimensional quantum system whose Hilbert space has dimension d , C is isomorphic to the set of $d \times d$ positive semidefinite matrices, whose normalized (i.e. unit-trace) members form the convex set of density matrices for the system, while the ambient linear space V is the space of $d \times d$ Hermitian matrices. We shall often use the abbreviation “PSD” for “positive semidefinite.”

3.1 Definition Let V, W be finite-dimensional real linear spaces equipped with cones $C \subset V$, $D \subset W$, and distinguished linear functionals $\lambda \in C^*$, $\tilde{\lambda} \in D^*$ that separate C, D from $-C, -D$ respectively. Let $\pi : V \rightarrow W$ be a linear map that takes C onto D (that is, $\pi(C) = D$), and maps the affine plane $L_\lambda := \{x \in V : \lambda(x) = 1\}$ onto the plane $M_{\tilde{\lambda}} := \{y \in W : \tilde{\lambda}(y) = 1\}$. An element (“state”) in $\hat{C} := L_\lambda \cap C$ is called *generalized unentangled* (GUE) relative to D if it is in the closure of the convex hull of the set of extreme points x of \hat{C} whose images $\pi(x)$ are extreme in $\hat{D} := D \cap M_{\tilde{\lambda}}$.

3.2 Definition We will call a pair of linear spaces V, W equipped with distinguished cones C, D , functionals $\lambda, \tilde{\lambda}$, and a map π , satisfying the conditions in the above definition, a *cone-pair*. As noted above, we write \hat{C}, \hat{D} for the normalized subsets of C, D , i.e. for $\{x \in C : \lambda(x) = 1\}$ and $\{x \in D : \tilde{\lambda}(x) = 1\}$. We will also sometimes call $\lambda, \tilde{\lambda}$ the *traces* on their respective cones, so that the condition on π above may be called *trace-preservation*. That is, with the usual physics terminology that extremal states are “pure” and nonextremal ones “mixed,” unentangled pure states of \hat{C} are those whose “reduced” states (images under π) are pure, and the notion extends to mixed states as in standard entanglement theory: unentangled mixed states in \hat{C} are those expressible as convex combinations of unentangled pure states (or limits of such combinations, though the latter is unnecessary in finite dimension).

It is easy to see that the motivating example of ordinary bipartite entanglement is a special case of this definition. Here, C is the cone of PSD operators on some

tensor product $A \otimes B$ of finite-dimensional Hilbert spaces, while D is the direct product of the cones of PSD operators on A and on B (intuitively, it is just the cone of all ordered pairs whose first member is a positive operator on A and whose second is one on B). λ is the trace. π is the map that takes an operator on $A \otimes B$ to the ordered pair of its “marginal” or “reduced” operators (“partial traces”) on A and B . Similarly, standard multipartite entanglement is a special case of GE. So we may view the GUE definition (in particular condition (a) of Definition 3.3 below) as based on extending the long-standing observation that for ordinary multipartite finite-dimensional quantum systems, a pure state is entangled if and only if at least one of its reduced density matrices is mixed.

It is perhaps mathematically more natural to define the *unnormalized* unentangled states of C relative to D , omitting all mention of $\lambda, \tilde{\lambda}$, and the normalization-preservation requirement on π . That is:

3.3 Definition Let C, D be cones in finite-dimensional real linear spaces V, W respectively, and let $\pi : V \rightarrow W$ map C onto D . $x \in C$ is *generalized unentangled* (relative to D, π) if either (a) x belongs to an extreme ray of C , and $\pi(x)$ belongs to an extreme ray of D , or (b) x is a positive linear combination of states satisfying (a), or a limit of such combinations.

It is easy to verify that the unnormalized GUE states are a (possibly non-generating, but otherwise regular) cone in V . If one introduces the notion of normalization in C via a functional λ , it is also easily verified that the normalized GUE states of Definition 3.1 are precisely the intersection of this cone with the normalization plane. (It is straightforward to introduce a normalization plane, and associated functional $\tilde{\lambda}$ on W , if desired, as the image of L_λ under π .)

Barnum *et al.* (2003b), and especially Barnum *et al.* (2003a), stressed applications in which the reduced state-set is obtained by selecting a distinguished subspace of the observables (Hermitian operators) on some quantum system. The reduced state-set is then the set of linear functionals (equivalently, consistent lists of expectation values for the distinguished observables) on this subspace of the space of all observables, that are induced by normalized quantum states². We dub this class of cone-pairs the *distinguished quantum observables* setting. Even in the more general cones setting, there is a natural notion of observables, and Definition 3.1 can be interpreted as restriction of the states to a subspace of the observables. To show this we employ a formalism of states, measurements, and observables that, in many variants, is frequently used as a touchstone of “operational” approaches to theories in the abstract³.

²It is worth noting that beyond the setting of standard quantum entanglement this is not in general a vacuous requirement: there can be normalized linear functionals on the reduced state set that are *not* obtainable by restriction from a quantum state on the set of all observables. Although all normalized functionals on the distinguished observables can be extended in many ways to normalized functionals on the full set, in some cases not all can be extended to *positive* functionals.

³By an “operational theory,” we mean one in which a theory describes various measure-

We view V^* as a space of real-valued observables. For $x \in V^*$ and $\eta \in \hat{C}$, we interpret $x[\eta]$ as the expectation value of observable x in state η . We view V as the dual of V^* in such a way that $x[\eta] = \eta[x]$ for all $x \in V^*, \eta \in V$. But what guarantee do we have that these expectation values behave in a reasonable way, as observables in an operational theory should? That is, can we view the expectation value $\eta(x)$ of an observable x in a state η as the expected value of some quantity being measured? By this we mean that x is associated with a quantity that takes different values depending on the outcome of the measurement, and the state determines the expectation value by determining probabilities for the different outcomes of the measurement, such that the value $\eta(x)$ is indeed the expectation value of the outcome-dependent quantity, calculated according to the probabilities assigned to the outcomes by the state.

We will only sketch the answer to this question; more details may be found in many places (though accompanied by additional concepts and formalism), notably Beltrametti and Bugajski (1997). In the structure we have described, of state-space and dual observable space, we are able to find a special class of observables, the “decision effects,” whose expectation value may be viewed as the probability of a measurement outcome. These “effects” are the elements of the initial interval $\mathcal{E} := [0, \lambda] \subset C^*$, i.e. the set of $x \in C^*$ satisfying $\lambda \geq_{C^*} x$. A (finite) *resolution of λ* is a set of effects $x_i \in \mathcal{E}$ such that $\sum_i x_i = \lambda$. For normalized states ω , it follows that $\omega(x_i) \geq 0$ and $\sum_i \omega(x_i) = 1$, so the values $\omega(x_i)$ may be viewed as probabilities of measurement outcomes, with a resolution of λ representing the mutually exclusive and exhaustive outcomes of some measurement. Then it can be shown that for *any* observable $A \in V^*$, a resolution \mathcal{R} of λ and an assignment of real values $v(x_i)$ to the outcomes x_i in \mathcal{R} can be found, such that for all normalized states ω , $\omega(A) = \sum_i \omega(x_i)v(x_i)$. For example, this is a consequence of (i) of Theorem 1 in Beltrametti and Bugajski (1997). In general the converse does not hold, giving rise to a generalization of observables sometimes known as *stochastic observables* for which not only does the analogous statement (which is (i) of Theorem 1 of Beltrametti and Bugajski (1997) where stochastic observables are just called observables) hold, but so does the converse of this analogue. The relation between the convex and the effect-algebras approach has been treated in various places (and aspects of it appear in some contexts, e.g. Ludwig (1983), even earlier than the formal notion of effect algebra). Some references are Gudder and Pulmannová (1998), Gudder *et. al.* (1999), Gudder (1999), and the book DallaChiara (2004) (especially Ch. 6). Barnum (2003a) explores the relation between probabilistic operational theories and “weak effect algebras,” as well as related more dynamical objects termed *operation algebras*, but without explicit consideration of observables. Bennett and Foulis (1997), Foulis *et al.* (1998), and Foulis (2000) address very closely related representational issues but without the constraint of convexity. The relations between convex and general effect-algebras and their representations are discussed in ?.

ments or operations one can perform on systems of the type described by the theory, and specifies a set of possible “states,” each of which determines the probabilities for the outcomes of all possible measurements, when the system is in that state.

We now show that our formalism of maps π onto cones D is equivalent to restriction to a subspace of observables.

3.4 Theorem

I) (“Observable restriction implies cone-pair”). Let C be a cone in V , and let $\lambda \in V^$ separate V from $-V$ (as in Def. 3.1), and let W^* be a subspace of V^* , containing λ . For $\eta \in V$, define $\eta| : W^* \rightarrow \mathbb{R}$ as the restriction of η to the subspace W^* , i.e. $\eta|(x) = \eta(x)$ for $x \in W^*$ and otherwise $\eta|(x)$ is undefined. Thus $\eta| \in (W^*)^* =: W$. Define $D = \{\eta| : \eta \in C\}$, $M_\lambda = \{y \in W : \lambda(y) = 1\}$. Define π as the restriction map $\pi := | : V \rightarrow W, \eta \mapsto \eta|$. Then $V, W, C, D, \lambda, \tilde{\lambda} (= \lambda), \pi$ form a cone-pair in the sense of Definition 3.2. That is, D is a cone in W , $\pi(C) = D$, and the image under $|$ of the plane $L_\lambda \equiv \{\eta \in V : \lambda(\eta) = 1\}$ is a translation of a plane separating D from $-D$.*

II) (“Cone-pair implies observable restriction”). Let $V, W, C, D, \lambda, \tilde{\lambda}, \pi$ be a cone-pair. Then there exists an injection (one-to-one map) $\tau : W^ \rightarrow V^*$, taking $\tilde{\lambda}$ to λ , such that π is the map from V to W that takes x to the function $x|_{W^*}$. Here $x|_{W^*}$ defined as the linear functional on W^* whose value on $a \in W^*$ is the value of x ’s restriction to $\tau(W^*)$ on $\tau(a)$.*

Remark concerning I: The restriction that the subspace W^* contain λ is hardly objectionable from an operational point of view. λ ’s expectation value is just the normalization constant, and is independent of which normalized state has been prepared. Therefore it can be measured without any resources, and there is no point in claiming that omitting it could represent a physically significant restriction on the means available to observe or manipulate a system.

Remark concerning II: The definition of $|$ in part I of the theorem involved a subspace W^* of V^* ; in part II we have defined W^* abstractly rather than as a subspace of V^* , so it is $\tau(W^*)$, which is isomorphic to W^* but *is* a subspace of V^* , to which we restrict states in defining $|$. (Of course, W^* *itself* is a subspace of V^* according to the category-theoretic definition of subspace.)

Proof: To prove part I, we must show that D is a cone in W , and λ separates it from $-D$. It is easy to verify linearity of $| \equiv \pi$ from the definition, and in finite dimensions, it is also easy to verify that linear maps from one vector space *onto* another (such as $|$) take cones to cones. For all $x \in \dot{C} =: C - \{0\}$, $\lambda[x] > 0$. But $\lambda[x] = x[\lambda]$ by duality, and by the definition of $|$ and the fact that $\lambda \in W^*$, $x[\lambda] = x|[\lambda] \equiv \lambda[x|]$, so $\lambda[x|] > 0$ for all $x \in \dot{C}$, i.e. (since $|$ maps \dot{C} onto \dot{D}), $\lambda[y] > 0$ for all $y \in D$. That is, λ separates D from $-D$.

To prove part II, let τ be π^* . That is, for all $x \in W^*$, $\eta \in V$, $\tau(x)[\eta] = x[\pi(\eta)]$. By duality, this gives $\eta[\tau(x)] = \pi(\eta)[x]$. Since, by Proposition 2.3, τ is an injection, this last equation determines $\pi(\eta)$ to be essentially $\eta|_{\tau(W^*)}$, as desired. The “essentially” refers to the fact that $\pi(\eta)$ is actually the pullback along τ

of this restriction; the two are the same function only if one identifies W^* with its image under τ . τ , in other words, tells us how W^* can be identified with a subspace of the full space V^* of observables, in such a way that $\pi(\eta)$ becomes identified with the restriction of η to W^* . \square

3.5 Proposition *In a cone-pair, π has the property that for $x \in \text{Extr } \hat{D}$, the set $\pi^{-1}(x)$ is convex, compact, and closed, and its extremal elements are extremal in \hat{C} .*

Proof: Convexity is immediate: if $y_1, y_2 \in C$, $\pi(y_1) = x$ and $\pi(y_2) = x$, then $\lambda y_1 + (1 - \lambda)y_2 \in C$ by convexity of C , and by linearity of π , $\pi(\lambda y_1 + (1 - \lambda)y_2) = \lambda(\pi(y_1)) + (1 - \lambda)\pi(y_2) = x$. Closedness of $\pi^{-1}(x)$ in the Euclidean metric topology follows from the fact that π , being a function from a finite-dimensional inner product space to a finite-dimensional normed space, is continuous (cf. e.g. Young, N., (1988), Exercise 7.3), and the preimage of a closed set under a continuous function is closed (cf. e.g. Kripke (1968), Corollary IV.C.4). Since finite intersections of closed sets are closed, $C \cap \pi^{-1}(x)$ is closed as well. Compactness follows from the fact that \hat{C} is compact (cf. e.g. Barvinok (2002)) hence a compact metric space, and a closed subset of a compact metric space is compact (Kripke (1968), Corollary VII.A.11).

Now let $x \in \text{Extr } \hat{D}$, and let $y \in \pi^{-1}(x) \cap C$ not be extremal in \hat{C} . We need to show that such a y is not extremal in $\pi^{-1}(x)$ either. $y \notin \text{Extr } \hat{C}$ means there are $y_1, y_2 \in \hat{C}$ with $y_1 \neq y_2$, $y = \lambda y_1 + (1 - \lambda)y_2$. By linearity of π , $x \equiv \pi(y) = \lambda\pi(y_1) + (1 - \lambda)\pi(y_2)$; since $x \in \text{Extr } (\hat{D})$, $\pi(y_1) = \pi(y_2) = x$. Hence $y_1, y_2 \in \pi^{-1}(x)$, so $y \notin \text{Extr } (\pi^{-1}(x) \cap C)$. \square

In important classes of examples, a stronger property holds:

3.6 Definition A cone-pair including C, D, λ, π is said to have the *unique preimage property* (UPIP) if $x \in \text{Extr } \hat{D}$ implies that $\pi^{-1}(x)$ consists of a single element (which must therefore be extremal).

Equivalently (because of Prop. 3.5), extremal reduced states have only extremal preimages.

3.7 Problem *Find nontrivial necessary and/or sufficient conditions (some are given below, but others almost certainly exist) for cone-pairs C, D, π to have the UPIP.*

Finally, note that the converse of the UPIP follows from Proposition 3.5 : If $\pi^{-1}(x)$ is unique, then it must be extremal, and x must be extremal as well.

4. GENERALIZED ENTANGLEMENT IN SPECIAL CLASSES OF CONES

We now formally define several “settings” in which to study GE; these are special classes of cone-pairs, physically and/or mathematically motivated.

4.1 Definition

- 1 *Distinguished quantum observables setting*, defined above. An equivalent formulation is the *Hermitian-closed (aka \dagger -closed) operator subspace setting*, in which the distinguished observable subspace is the Hermitian operators belonging to a \dagger -closed subspace, containing the identity operator, of the complex vector space of all linear operators on a quantum system.
- 2 *Lie-algebraic setting*. Here, C is the cone of positive Hermitian operators on a (finite-dimensional) Hilbert space carrying a Hermitian-closed Lie algebra \mathfrak{g} (playing the role of W^*) of Hermitian operators (with Lie bracket $[X, Y] := i(XY - YX)$, and containing the identity operator) and D the cone (in $(W^*)^* =: W$) of linear functionals on \mathfrak{g} induced from positive Hermitian elements of C by restriction to W^* .
- 3 *Associative algebraic setting*. Here, the distinguished observables are the Hermitian elements of some associative subalgebra of the associative algebra of all operators on a quantum system.

By construction, the Lie-algebraic and associative algebraic settings are special cases of the distinguished quantum observables case. As noted in Barnum *et al.* (2003b), since the Lie-algebraic setting was defined to involve finite-dimensional \dagger -closed matrix representations, the Lie algebras involved are necessarily reductive i.e., the direct product⁴ of a semisimple and an Abelian part.

A distinction that can be nontrivially made within all the settings in the above list is between those in which the distinguished observables act *irreducibly*, and those in which there is a nontrivial subspace invariant under the action of all observables.

4.2 Proposition *In the \dagger -closed operator subspace setting, the distinguished subspace has a basis of Hermitian operators that is orthonormal in the trace inner product $\langle A, B \rangle = \text{tr } AB$.*

Because of this proposition, we may construct an orthogonal projection operator (some would call it a superoperator) Π_S , acting on the space of Hermitian operators by projecting into the subspace of distinguished observables. We can also use such a basis to define a measure of entanglement for pure states, the *relative purity* (although the name may be slightly misleading, for reasons we will explain).

4.3 Definition Let ω be a state on a \dagger -closed set S of quantum observables.

⁴As Lie algebras; the induced direct product of the algebras considered as vector spaces (i.e. without their Lie bracket structure) is also a vector space direct sum.

The *purity* $P(\omega)$ of a state ω is defined by letting X_α be an orthonormal (in trace inner product) basis of S . Then

$$P(\omega) := \sum_{\alpha} (\omega(X_\alpha))^2. \quad (2)$$

Note that any state ω on the *full* operator space corresponds uniquely to a density operator ρ_ω , defined by the condition $\text{tr}(\rho_\omega X) = \omega(X)$ for all observables X .

Closely related to the above purity is the *relative purity* of a pure state $|\psi\rangle$ of the overall quantum system; this is defined equal to the purity of the state it induces on S , or equivalently, with X_α as above,

$$P_S(|\psi\rangle) := \sum_{\alpha \in S} |\langle \psi | X_\alpha | \psi \rangle|^2. \quad (3)$$

In fact, this definition could be straightforwardly extended to mixed states ω on the full Hilbert space, as

$$P_S(\omega) := \sum_{\alpha \in S} |\text{tr} \omega X_\alpha|^2. \quad (4)$$

However, a requirement for entanglement measures is convexity (Vidal, 2000), and the above extension lacks this as well as other desirable properties. We will generally extend pure-state entanglement measures μ to mixed states via the *convex hull (often called convex roof) construction* (cf. e.g. Uhlmann (1998); Bennett *et al.* (1996a)) standard in ordinary entanglement theory: the value of the measure μ on a mixed state ω is the infimum, over convex decompositions $\omega = \sum_i p_i \pi_i$ of ω into pure states π_i , of the average value of the pure-state measure, that is, of $\sum_i p_i \mu(\pi_i)$. This is convex by construction.

Defining Π_S as the projection superoperator onto the operator subspace S , it is easily verified that

$$P_S(\omega) := \sum_{\alpha} |\text{tr} \Pi_S(\rho_\omega) X_\alpha|^2 \equiv \text{tr} [\Pi_S(\rho_\omega)^2]. \quad (5)$$

For any density operator ρ , we call $\Pi_S(\rho)$ the associated *reduced* density operator; note that it need *not* be a positive operator on the full state space (although it is in the standard multipartite case). This is not problematic because for any PSD element R of the *distinguished* observable space, $\text{tr} \Pi_S(\rho) R \geq 0$, of course.

The following proposition is immediate from Theorem 14 of Barnum *et al.* (2003b).

4.4 Proposition *In the irreducible Lie-algebraic setting, pure states with maximal relative purity are generalized unentangled.*

The converse is *not* true in general. Also, the analogue of Prop. 4.4 for the general Lie-algebraic setting (allowing *reducible* representations) can be shown by example to be false.

Another situation in which maximal relative purity implies generalized unentanglement is embodied in the following.

4.5 Proposition *In the \dagger -closed operator subspace setting states with unit relative purity have unique preimages, and are therefore generalized unentangled.*

Proof: A necessary and sufficient condition for a normalized state ω on the space of all observables to be pure is $\text{tr}(\rho_\omega^2) = 1$. (Henceforth we suppress the ω -dependence of ρ .) Letting X_α be an orthonormal basis for the space of all observables such that a subset (denoted by the letter β for the index) indexes the distinguished subspace S , with another subset (indexed by γ) indexing S^\perp , and writing $\langle X_\alpha \rangle$ for $\text{tr} \rho X_\alpha$, we have $\rho = \sum_\alpha \langle X_\alpha \rangle X_\alpha$. From this and orthonormality of the X_α it is easy to see that $\text{tr}(\rho^2) = \sum_\alpha \langle X_\alpha \rangle^2$. $P_S(\rho) \equiv \sum_{\beta \in S} \langle X_\beta \rangle^2$; since extremal overall states have $\text{tr}(\rho^2) = 1$, $P_S(\rho)$ for a pure state ρ can never be greater than 1, since it is a sum of a subset of the positive quantities $\langle X_\alpha \rangle^2$ which sum to 1. Let X have unit relative purity, i.e. $\sum_{\beta \in S} X_\beta^2 = 1$. This implies $\sum_{\gamma \in S^\perp} \langle Y_\gamma \rangle^2 = 0$, which requires $\langle X_\gamma \rangle = 0$ for all $\gamma \in S^\perp$. Thus, $P_S(\rho)$ has a unique preimage, namely itself, so $\rho = P_S(\rho)$. If $P_S(\rho)$ did not induce an extremal state in the convex set of reduced states, it would be a convex combination of distinct operators ρ_1 and ρ_2 inducing distinct reduced states; these would have distinct preimages, but the convex combination of these preimages be $P_S(\rho) \equiv \rho$, violating the assumption that ω is pure. \square

What about states whose relative purity is maximal among all states, even when this maximal value is not unity? When the maximum is not unity, no pure state has an unchanged reduced density matrix: all pure state density matrices project to reduced “density matrices” that are either mixed, or not even PSD. Thus we cannot immediately conclude that $\sum_\beta \langle X_\beta \rangle^2 = 1$, so we do not have $\langle X_\gamma \rangle = 0$ for all $X_\gamma \in S^\perp$. If there is nevertheless a unique preimage, i.e. the X_γ are uniquely determined by the X_β (for β indexing S), it must be a consequence of positive semidefiniteness of the initial state, since linear algebra alone gives no restrictions on the X_γ . However, because relative purity is a strictly convex function of the reduced density matrix, a state’s having maximal, even if not unit, relative purity, implies generalized unentanglement in the \dagger -closed operator subspace framework. It does not, however, imply the other part of Proposition 4.5, that the reduced state has a unique preimage. Formally:

4.6 Proposition *Let $x \in \hat{C}$ be such that the relative purity of x is no less than that of every other element of \hat{C} . Then x is generalized unentangled.*

Proof: The relative purity of ω is just the Euclidean norm of $\Pi_S(\rho_\omega)$ (with respect to the trace inner product). Suppose ω has maximal relative purity, i.e.

$\|\Pi_S(\rho_\sigma)\| \leq \|\Pi_S(\rho_\omega)\|$ for all $\sigma \in \hat{C}$. Suppose there are $\alpha, \beta \in \hat{D}, \alpha \neq \beta$, such that $\Pi_S(\rho_\omega) = \mu\rho_\alpha + (1 - \mu)\rho_\beta$. Then by the triangle inequality $\|\Pi_S(\rho_\omega)\| \leq \|\mu\rho_\alpha\| + \|(1 - \mu)\rho_\beta\| = \mu\|\rho_\alpha\| + (1 - \mu)\|\rho_\beta\|$. Since neither $\|\rho_\alpha\|$ nor $\|\rho_\beta\|$ is greater than $\|\Pi_S(\rho_\omega)\|$, we must have $\|\rho_\alpha\| = \|\rho_\omega\| = \|\Pi_S(\rho_\omega)\|$, so there is equality in the triangle inequality. That requires $\mu\rho_\alpha$ to be proportional to $(1 - \mu)\rho_\beta$, however, so that $\rho_\alpha = \rho_\beta = \Pi_S(\rho_\omega)$. This shows that $\Pi_S(\rho_\omega)$ is extremal in the set of reduced density operators corresponding to states in \hat{D} . In other words, ω is generalized unentangled. \square

It follows from the representation theory of associative algebras that the UPIP holds for the irreducible associative algebraic setting. The other case in which we know it holds is the irreducible semisimple Lie algebraic setting. In this setting, the observables consist of the Hermitian part (itself a real Lie algebra) of a complex Lie algebra represented faithfully and irreducibly by matrices acting on a finite-dimensional complex Hilbert space, and including the identity matrix I . Such Hermitian parts of irreducible matrix Lie algebras are precisely the real semisimple algebras possibly extended by the identity. The identity is relatively unimportant since all normalized states have the same value on it: the normalization condition is the affine plane $\omega(I) = 1$, so the convex structure of the state space is entirely determined by the expectation values of the traceless operators. We introduce a bit more notation in order to state a result, proved in Barnum *et al.* (2003b), that includes this and other important facts about the irreducible Lie-algebraic case.

A *real* Lie algebra of Hermitian operators may be thought of as a distinguished family of Hamiltonians, which generate (via $h \mapsto e^{ih}$) a Lie group of unitary operators, describing a distinguished class of reversible quantum dynamics. More generally, we might want Lie-algebraically distinguished completely positive (CP) maps, $\rho \mapsto \sum_i A_i \rho A_i^\dagger$ describing open-system quantum dynamics. We call the operators A_i the “Hellwig-Kraus” or “HK” operators, since they appear to have been introduced in Hellwig and Kraus (1969, 1970) (see also Kraus (1983); Choi (1975)). The HK operators for a given CP map are not unique, but this does not lead to nonuniqueness of any of the objects we define in terms of them below. A natural Lie-algebraic class of CP-maps has HK operators A_i in the topological closure $e^{\mathfrak{h}_c \oplus \mathbf{1}}$ of the Lie group generated by the *complex* Lie algebra $\mathfrak{h}_c \oplus \mathbf{1}$ ⁵. Having HK operators in a group ensures closure under composition. Using $\mathfrak{h}_c \oplus \mathbf{1}$ allows non-unitary HK operators. Topological closure introduces singular operators such as projectors.

Define an \mathfrak{h} -state to be a linear functional on a *complex* matrix Lie algebra \mathfrak{h} belonging to the convex set of such states induced by normalized quantum states on the full representation space. Complex-linearity ensures that the convex structure of such a state space is the same as that of the states induced by

⁵ \mathfrak{h}_c is constructed by taking the complex linear span of a basis for \mathfrak{h} . $\mathfrak{h}_c \oplus \mathbf{1}$ guarantees inclusion of the identity operator $\mathbf{1}$.

taking as the distinguished observables only the Hermitian elements (a real Lie algebra we denote $\text{Re}(\mathfrak{h})$), which is the definition we used above for the Lie-algebraic setting.

4.7 Theorem *Let \mathfrak{h} be a complex irreducible matrix Lie algebra, with \mathfrak{h}_o its traceless (semisimple) part and $\text{Re}(\mathfrak{h})$ its Hermitian part. The following are equivalent for a density matrix ρ inducing the \mathfrak{h} -state λ :*

- (1) λ is a pure \mathfrak{h} -state, that is, it is extremal in the convex set of normalized linear functionals on \mathfrak{h} .
- (2) $\rho = |\psi\rangle\langle\psi|$ with $|\psi\rangle$ the unique ground state of some H in $\text{Re}(\mathfrak{h})$.
- (3) $\rho = |\psi\rangle\langle\psi|$ with $|\psi\rangle$ a minimum-weight vector (for some simple root system of some Cartan subalgebra) of \mathfrak{h}_o .
- (4) λ has maximum purity relative to the subspace $\text{Re}(\mathfrak{h})$ of observables.
- (5) ρ is a one-dimensional projector in the topological closure of $\overline{e^{\mathfrak{h}}}$.

4.8 Problem *Does the implication from GUE to maximal relative purity, hold in other natural situations?*

As already noted it is fairly easy to show by example that in the Lie-algebraic setting but without the assumption of irreducibility, the UPIP need *not* hold. A more general question suggests itself:

4.9 Problem *In the \dagger -closed operator subspace setting, does the UPIP hold whenever the distinguished operators act irreducibly?*

5. ANALOGUES OF LOCAL MAPS

Our work on GE raises many questions arising from the closely related problems of finding natural generalizations or analogues of the notions of LOCC (*Local Operations and Classical Communication*) and of monotone entanglement measures (or *entanglement monotones* (Vidal, 2000)). The relation comes from requiring that a reasonable entanglement measure be nonincreasing under LOCC operations; if one found a natural generalization of this notion of LOCC to our more general settings, it would also be natural to look for measures of GE monotone under this generalization. Here, we briefly present some ideas from Barnum *et al.* (2003b) (with a few minor extensions) on how to generalize LOCC; that paper contains more on this topic and on GE measures. Some of the most fundamental questions remain open, so we will concentrate on sketching the situation in hopes of stimulating further work.

The semigroup of LOCC maps, introduced in Bennett *et al.* (1996b), and the preordering it induces on states according to whether or not a given state can be transformed to another by an LOCC operation are at the core of entanglement theory. LOCC maps are precisely those implementable by using CP quantum maps on the local subsystems, and classical communication, e.g. of “measure-

ment results,” between systems. We now formalize this notion, beginning with the notion of *explicitly decomposed* map which, however, can apply to the general case, not just the quantum one. An *explicitly decomposed* trace-preserving map $\{M_k\}_{k \in K}$ is a set of maps M_k that sum to a trace-preserving one M . The *conditional composition* of an explicitly decomposed map $\{M_k\}_{k \in K}$ with a set of explicitly decomposed maps $N_k := \{N_{nk}\}_{n \in N_k}$ is the explicitly decomposed map $\{N_{nk} \circ M_k\}_{k \in K, n \in N_k}$. We can view each M_k as being associated with measurement outcome k , obtained (given a state ω) with probability $\text{tr } M_k(\omega)$, and leading to the state $M_k(\omega)$ when outcome k is obtained. The conditional composition of $\{M_k\}_{k \in K}$ and $\{N_{nk}\}_{n \in N_k}$ can be implemented by first applying M and then, given measurement outcome k , applying N_k . There are analogous definitions of explicitly decomposed maps and conditional composition without the trace-preservation condition.

In the usual quantum case, closing the set of one-party (aka *unilocal*) maps (for all parties) under conditional composition gives the LOCC maps. The semigroup generated by composition of unilocal explicitly decomposed maps having a single HK operator in their decomposition, is often known as SLOCC (for *stochastic* LOCC). SLOCC involves local quantum measurements and classical communication conditional on a *single* sequence of local measurement results, when each local measurement is performed in a manner that preserves all pure states (i.e., with a single HK operator for each outcome). Its mathematical structure is relatively simple, as the part generated by nonsingular HK operators is the trace-nonincreasing part of a representation of a product of various $\text{GL}(d_i)$, with the factors acting on local systems of dimension d_i ⁶.

When the distinguished observables form a semisimple Lie algebra \mathfrak{h} , a natural multipartite structure can be exploited to generalize LOCC, as well as the larger, more tractable class of *separable* maps; see Barnum *et al.* (2003a,b). In generalizing LOCC to the convex setting, two aspects of LOCC must be considered: first, that it constrains maps to be *completely positive*; second, that it also constrains them to have certain *locality* properties.

A *positive* map of D is a linear map $A : V \rightarrow V$ such that $A(D) \subseteq D$. The map A is *trace preserving* if $\text{tr}(x) = \text{tr}(A(x))$ for all x . This definition corresponds to positive, but not necessarily CP, maps in quantum settings. Without additional algebraic structure, it is not possible to define a unique “tensor product” of cones, as would be required to distinguish between positive and CP maps (Namioka and Phelps (1969); Wittstock (1974) (cited in Wilce (1992))). In a continuum of possible products of cones, there are two natural possibilities that are in a sense the two extremes. The first is the convex closure of the set of tensor-products of the cones’ vectors, which for the case of the product of two quantum systems’ unnormalized state spaces gives the separable states of the bipartite system. The second is to use the dual cone of the cone obtained by

⁶We are not certain if the full LOCC semigroup is the trace-nonincreasing part of the topological closure of this representation, but it seems a reasonable possibility.

applying the first construction to the duals of the cones; in the quantum case, it gives the set of (unnormalized) states that are positive on product effects (this is isomorphic to the cone of positive but not CP operators between the state spaces, by the “Choi-Jamiolkowski” isomorphism between $V \otimes V$ and $\mathcal{L}(V)$). It is not clear how to pick out a natural case between these extremes in general without adding algebraic structure, except perhaps if the cones are self-dual with respect to non-degenerate inner products on the real vector spaces. In that case, one could pick a self-dual cone between the two constructions (which would give the usual state space of a bipartite system in the quantum case).

The family of positive maps of C is closed under positive combinations and hence forms a cone. In the Lie-algebraic, or even the bipartite setting, the extreme points of this cone are not easy to characterize (see, for example, Wilce (1992), p. 1927, Gurvits (2002)). We seek generalizations of the notion of complete positivity to the cones setting. We might explicitly introduce a cone representing the “tensor product” extension of D and require extendibility or “liftability” of the map to D . Another, perhaps more uniquely determined, approach might begin from the observation that the extreme points of the cone of completely positive maps are extremality preserving: for all extremal (belonging to an extreme ray) $x \in D$, $A(x)$ is extremal. However there are extremality preserving positive, not CP, maps. An example is partial transposition for density operators of qubits. In Barnum *et al.* (2003b) we explore how one might rule these out. There is also the question of why extremality preservation would be a natural physical or operational, as opposed to mathematical, requirement.

To try to generalize the notion of locality, we introduce the idea of *liftability*. We say that a positive map A on D can be lifted to C if A preserves the nullspace of π , or, equivalently, if there exists a positive map A' on C such that $\pi(A(x)) = A'(\pi(x))$. In this case, we say that A' is the lifting of A to C .

In standard multipartite quantum entanglement, *unilocal* maps (ones that act nontrivially only on one factor) are liftable to the cone of local observables; they have a well-defined action there. But so are tensor product maps $\mathcal{A} \otimes \mathcal{B} \otimes \cdots \mathcal{Z}$, and in the case when some of the subsystems are of the same dimension, so are maps performing permutations among the isodimensional factors. To get LOCC we would need to rule out the latter two cases, leaving the unilocal maps; then one can generate a semigroup from the unilocal maps by conditional composition of explicitly decomposed trace-preserving maps. On the other hand, in the standard quantum case the semigroup of maps generated by conditional composition of maps liftable to the distinguished subcone might enjoy many of the same properties of the usual LOCC maps, so it may be worth study in the general setting.

5.1 Problem *Is the semigroup generated by completely positive unilocal quantum maps and pairwise exchanges of isodimensional systems the full semigroup generated by conditional composition of liftable-to-local-observables explicitly de-*

composed maps?

Note that using liftability to define locality may be of some help in ruling out local non-completely positive maps, since all maps must be positive on the overall cone. It is especially helpful if the answer to Problem 5.1 is “yes.” When no subsystem has dimension greater than the square root of the overall dimension, it is then fully effective in imposing complete positivity, because for any local map M , complete positivity of M is equivalent to positivity of the unilocal map $\text{id} \otimes M$ where the identity map id acts on a Hilbert space at least as large as the one M acts on.

In the standard multipartite quantum case, the high degeneracy of unilocal operators can also be used to help distinguish them in a way not so directly dependent on explicit introduction of cones to represent individual systems—and similarly one can use spectral information about HK operators to characterize ones that act on the *same* single system, thereby characterizing LOCC in terms of conditional composition of explicitly decomposed maps whose HK operators together satisfy certain spectral conditions (Barnum *et al.*, 2003b). However, it is not clear how to abstract this to general cones. Perhaps something can be done with the facial structure of the cone D , or of the cone of positive maps on D (or of other subcones of maps chosen as abstractions capturing aspects of complete positivity). A more in-depth investigation of dynamics generalizing LOCC thus remains as a challenging and many-faceted area for research, as does the investigation of measures of GE nonincreasing under such maps.

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6. * References

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